Intergrerend project dynamische systemen

1 April 2015

university of groningen

Exam Problem Sheet

The exam consists of 4 problems. You may answer in Dutch or in English. You have 120 minutes to answer the questions. Give brief but precise answers. You can achieve 60 points in total which includes a bonus of 5 points.

1. **[5+5+5 Points.]**

   For each of the following bifurcations of equilibrium points of time continuous systems, plot the bifurcation diagram and describe in words the bifurcation scenario, and give an explicit example (i.e. a one-parameter family of systems showing the respective bifurcation).

   (a) Transcritical bifurcation.

   (b) Pitchfork bifurcation.

   (c) Hopf bifurcation.

2. **[2+2+2+4+5 Points.]**

   Consider the planar system

   $$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

   where a is a positive constant.

   (a) Show from the eigenvalues that the equilibrium at the origin is asymptotically stable.

   (b) Give the (real) canonical form for the system (note that you might have to distinguish between different cases depending on the size of a).

   (c) Show that $L(x, y) = x^2 + y^2$ is a Lyapunov function for the system.

   (d) State Lasalle's Invariance Principle.

   (e) Use Lasalle's Invariance Principle to prove again that the equilibrium at the origin is asymptotically stable and that the basin of attraction is the full plane. (Hint: consider disks of arbitrary radius centered at the origin.)

— please turn over —

3. **[10 Points.]**

Consider the family of one-dimensional systems $x' = f(x, a)$ with parameter $a \in \mathbb{R}$. Suppose that for $x_0, a_0 \in \mathbb{R}$,

(i) $f(x_0, a_0) = 0$,

(ii) $\frac{\partial f}{\partial x}(x_0, a_0) = 0$,

(iii) $\frac{\partial^2 f}{\partial x^2}(x_0, a_0) \neq 0$, and

(iv) $\frac{\partial f}{\partial a}(x_0, a_0) \neq 0$.

Show that the systems has a saddle-node bifurcation at (x_0, a_0).

4. **[6+9 Points.]**

(a) State the definition of chaos for a discrete time system.

(b) Let $\Sigma = \{(s_0, s_1, s_2, \ldots) : s_k \in \{0, 1\}\}$ be the space of half-infinite sequences of the symbols 0 and 1. The map $d : \Sigma \times \Sigma \to \mathbb{R}$ which maps $s = (s_0, s_1, s_2, \ldots)$ and $t = (t_0, t_1, t_2, \ldots)$ to

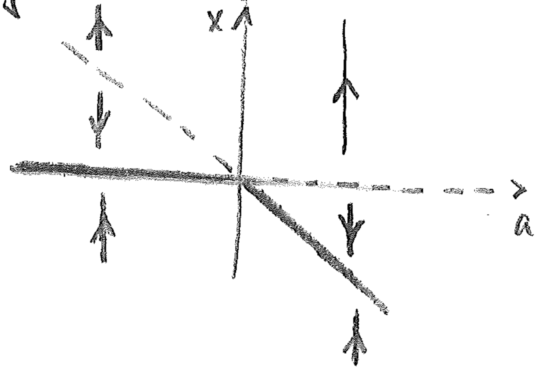
$$d(s, t) = \sum_{k=0}^{\infty} \frac{|s_k - t_k|}{2^k}$$

defines a metric on Σ. Argue that the shift map

$$\sigma : \Sigma \to \Sigma, \quad s = (s_0, s_1, s_2, \ldots) \mapsto \sigma(s) = (s_1, s_2, s_3, \ldots)$$

is chaotic.

1. (a) bifurcation diagram:



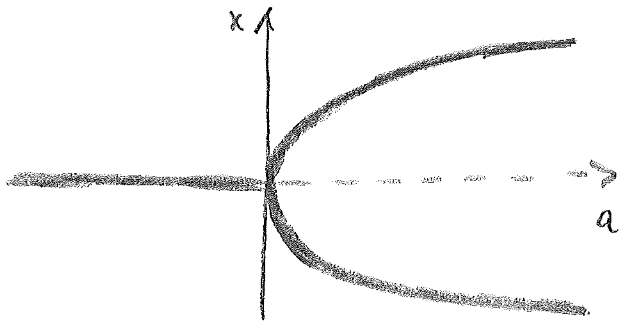
"two equilibria cross each other and exchange their stability"

bold: stable

dashed: unstable

example: $x' = ax \pm x^2$

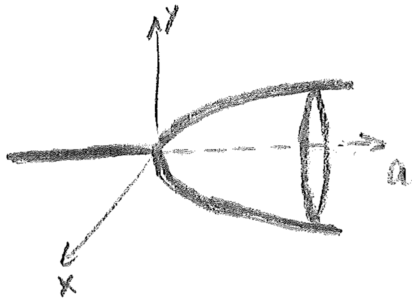
(b) bifurcation diagram



"a (central) equilibrium changes stability and gives birth to two new equilibria of opposite stability"

example: $x' = ax \mp x^3$

(c)



"a central equilibrium changes stability and a periodic orbit (limit cycle) of opposite stability is growing out of the equilibrium"

example: in polar coordinates

$$r' = ar - r^3$$
$$\theta' = 1$$

2. $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & -a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} =: A \begin{pmatrix} x \\ y \end{pmatrix}$ $\qquad a > 0$

(a) eigenvalues:

$$\det(A - \lambda\, id) = -\lambda(-a-\lambda) + 1 \overset{!}{=} 0$$

$$\Leftrightarrow \quad \lambda(a+\lambda) = -1$$

$$\Leftrightarrow \quad \lambda_\pm = -\frac{a}{2} \pm \sqrt{\frac{a^2}{4} - 1}$$

$$\Rightarrow \quad \mathrm{Re}\,\lambda_\pm < 0$$

$\Rightarrow$ equilibrium at origin is asymptotically stable

(b) for $a \geq 2$, eigenvalues are real.

The canonical form is

$$\begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix} = \begin{pmatrix} -\frac{a}{2} + \sqrt{\frac{a^2}{4} - 1} & 0 \\ 0 & -\frac{a}{2} - \sqrt{\frac{a^2}{4} - 1} \end{pmatrix}$$

for $0 < a < 2$, eigenvalues are complex (not real).

The canonical form is

$$\begin{pmatrix} \mathrm{Re}\,\lambda_+ & \mathrm{Im}\,\lambda_+ \\ -\mathrm{Im}\,\lambda_+ & \mathrm{Re}\,\lambda_+ \end{pmatrix} = \begin{pmatrix} -\frac{a}{2} & \sqrt{1 - \frac{a^2}{4}} \\ -\sqrt{1 - \frac{a^2}{4}} & -\frac{a}{2} \end{pmatrix}$$

(c) $\dot{L}(x,y) = L_x x' + L_y y' = 2x(y) + 2y(-x-ay)$

$$= -2ay^2 \leq 0$$

as also $L(x,y) \geq 0$ and $L(x,y) = 0 \Leftrightarrow (x,y) = (0,0)$

L is a Lyapunov function.

(d) X^* equilibrium point of $X' = F(X)$,
U open neighbourhood of X^*, $L : U \to \mathbb{R}$
Lyapunov function, $P \subset U$ compact neighb.
of X^*, and there is no entire solution in $P \setminus \{X^*\}$
on which L is constant. Then X^* is asymptotically
stable and P is contained in the basin of
attraction of X^*.

(e) Let $P = B_r(0)$ where $B_r(0)$ is the closed
ball of radius $r > 0$ centered at the origin.
P is positively invariant as can be seen
from the Lyapunov function defined in
part (c).

To be shown: P contains no solutions
apart from the equilibrium at the origin
along which the Lyapunov function is constant.

We have: let $(x(t), y(t))$ be a solution

$$\Rightarrow \quad \frac{d}{dt} L(x(t), y(t)) = -a\, y(t)^2 \qquad (\text{see (c)})$$

$$\Rightarrow \quad y(t) \equiv 0$$

$$\Rightarrow \quad x'(t) \equiv y(t) \equiv 0$$

$$\Rightarrow \quad x(t) \equiv \text{constant}$$

But the only possible solution $(x(t), y(t)) \equiv (c, 0)$
is the one which has $c = 0$, i.e. the equil. sol.

3. $f(x_0, a_0) = 0, \quad \frac{\partial f}{\partial a}(x_0, a_0) \neq 0$

$\Rightarrow \quad \exists \varepsilon > 0$ and function

$(x_0 - \varepsilon, x_0 + \varepsilon) \to \mathbb{R}, \quad x \mapsto a(x)$

such that $\quad f(x, a(x)) = 0$

(by implicit function theorem)

We have

$$a'(x_0) = \frac{-\frac{\partial f}{\partial x}(x_0, a_0)}{\frac{\partial f}{\partial a}(x_0, a_0)} = 0 \qquad \text{since} \quad \frac{\partial f}{\partial x}(x_0, a_0) = 0 \text{ and } \frac{\partial f}{\partial a}(x_0, a_0) \neq 0$$

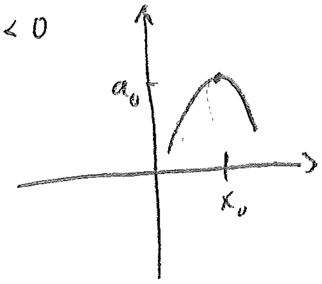
$$a''(x_0) = \frac{d}{dx}\bigg|_{x = x_0} \frac{-\frac{\partial f}{\partial x}(x, a(x))}{\frac{\partial f}{\partial a}(x, a(x))}$$

$$= - \frac{\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial a \partial x} a'\right)\frac{\partial f}{\partial a} - \frac{\partial f}{\partial x}\left(\frac{\partial^2 f}{\partial x \partial a} + \frac{\partial^2 f}{\partial a^2} a'\right)}{\left(\frac{\partial f}{\partial a}\right)^2}\Bigg|_{x = x_0}$$

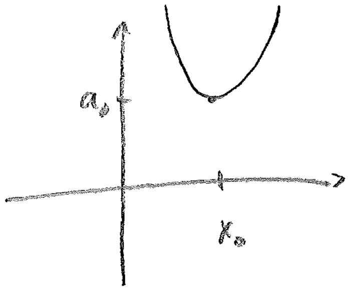
$$= - \frac{\frac{\partial^2 f}{\partial x^2}(x_0, a_0)}{\frac{\partial f}{\partial a}(x_0, a_0)} \neq 0$$

$\Rightarrow$ graph of implicit function

$a'' < 0$



$a'' > 0$



reflecting over the diagonal give the bifurcation diagrams

4. (a) The system $x_{n+1} = f(x_n)$, $n = 0, 1, 2, \ldots$,

with $f: \Lambda \to \Lambda$ is called chaotic

if (i) periodic points are dense

(ii) f is transitive, i.e. for any open intervals U and V there is $n > 0$ such that $f^n(U) \cap V \neq \emptyset$

(iii) f has sensitive dependence on initial conditions, i.e. there is $\beta > 0$ such that for any $x_0 \in \Lambda$ and any neighbourhood U of x_0 there is $y_0 \in U$ and $n > 0$ such that $d(f^n(x_0), f^n(y_0)) > \beta$

where $d(\cdot,\cdot)$ is a metric.

(b) - σ has property (i):

Let $t \in \Sigma$ and $\varepsilon > 0$.

Choose $n \in \mathbb{N}$ such that $\frac{1}{2^n} < \varepsilon$

set $s = (t_0, t_1, \ldots, t_n, t_0, t_1, \ldots, t_n, t_0, \ldots)$

$\Rightarrow$ s periodic and

$d(s, t) < \varepsilon$

- σ has property (ii):

let $s^* = (\underbrace{0, 1}_{\text{all blocks of length 1}}, \underbrace{0,0,0,1,1,0,1,1}_{\text{all blocks of length 2}}, \underbrace{\quad\quad}_{\text{all blocks of length 3}}, \ldots)$

$\Rightarrow \quad \sigma^k s^*, \ k = 0,1,2,\dots$ dense in Σ.

This implies transitivity

- σ has property (iii):

Set $\beta = 2$. Let $\varepsilon > 0$ and $s \in \Sigma$.

Choose n such that $\frac{1}{2^n} < \varepsilon$.

Set $\quad t = (s_0, s_1, \dots, s_n, \hat{s}_{n+1}, \hat{s}_{n+2}, \dots)$

where $\quad \hat{s}_j = \begin{cases} 1 & \text{if } s_j = 0 \\ 0 & \text{if } s_j = 1 \end{cases}$

$\Rightarrow \quad d(s,t) < \varepsilon$ and

$d(\sigma^{n+1}(s), \sigma^{n+1}(t)) = 2$